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Forced convection in a helical pipe filled with a saturated porous medium

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Abstract

An analysis is made of laminar forced convection in a helical pipe of circular cross-section and filled by a porous medium saturated with a fluid, for the case when the curvature and torsion of the pipe are both small. The Darcy model is employed, and boundaries with either uniform flux or uniform temperature are considered. It is found that curvature induces a secondary flow at first order in the parameter $\varepsilon = \kappa a$, where κ is the curvature and a is the radius of the pipe. On the other hand, the Nusselt number is unchanged to first order in ε but is increased at second order, for either set of thermal boundary conditions. The effect of torsion on the velocity appears at second order, but torsion does not affect the Nusselt number at second order.

1. Introduction

The hydrodynamics of laminar flow of a Newtonian fluid in a helical pipe has received considerable attention. Applications have included blood flow in a segment of human coronary artery [1]. However, the heat transfer aspect has received less attention, and we located fewer than a dozen papers [2–12] on that aspect. In the case of a porous medium occupying the helical pipe, it appears that nothing has been published, on either the hydrodynamic or heat transfer aspect, and the present paper is aimed to provide an analysis to fill that gap. The key to progress towards an analytical solution is to employ the orthogonal set of coordinates introduced by Germano [13,14]. Our main results are based on a perturbation expansion for small values of the curvature and torsion.

It is envisaged that our work might be applicable to flow in a section of a coronary artery blocked by some porous tissue. No experimental data is available for comparison.

2. Analysis

On the Darcy model the governing equations for mass conservation and momentum are

$$\nabla \cdot \mathbf{v}^* = 0, \tag{1}$$

$$\frac{\mu}{K}\mathbf{v}^* = -\nabla P^*. \tag{2}$$

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Nomenclature

a	radius of the helical pipe, defined in Fig. 1b
A	constant defined in Eq. (52)
c_p	specific heat at constant pressure
Da	Darcy number, K/a^2
F	function defined in Eq. (56)
J_α	Bessel functions of the first kind
K	permeability of the porous medium
Nu	Nusselt number, defined in Eq. (38)
p	pitch, defined in Fig. 1a
P	dimensionless pressure, $P^*/\rho U^2$
P^*	pressure
q''	wall heat flux
r	dimensionless radial coordinate, r^*/a
r^*	radial coordinate
R	radius of the coil, defined in Fig. 1a
Re	Reynolds number, Ua/ν
s	dimensionless axial coordinate, s^*/a
s^*	axial coordinate
T^*	temperature
$\widehat{T}$	dimensionless temperature, defined in Eq. (37)
T^*_{m}	bulk mean temperature, defined in Eq. (39)
T^*_{w}	wall temperature
u, v, w	dimensionless velocity components, u^*/U, $v^*/U, w^*/U$
u^*,v^*,w^*	velocity components
U	characteristic velocity

Greek symbols

ε	dimensionless curvature, κa
θ	angle, defined in Fig. 1b
κ	curvature, $R/(R^2+p^2)$
λ	ratio of torsion to curvature, τ/κ
ν	fluid kinematic viscosity
ξ	angle, defined in Eq. (10)
ρ	fluid density
τ	torsion, $p/(R^2+p^2)$
ϕ	angle, defined in Fig. 1b, $-\int_{s_0}^{s}\tau\,\mathrm{d}s'$
ω	parameter defined in Eq. (9)

Here $\mathbf{v}^* = (u^*, v^*, w^*)$ is the Darcy (filtration) velocity, P^* is the pressure, μ is the fluid viscosity and K is the permeability of the porous medium. The Darcy momentum Eq. (2) applies on the scale of a representative elementary volume (REV) of the porous medium for slow flow (pore-scale Reynolds number less than unity). Curvature and torsion affect the inertial terms in the momentum equation, but on the Darcy model inertial terms are neglected anyhow.

The following dimensionless variables are introduced and fitted to Germano's helical orthogonal coordinate system (see Fig. 1a and b for the definition of the coordinates).

$$s=\frac{s^*}{a},\quad r=\frac{r^*}{a},\quad (u,v,w)=\left(\frac{u^*}{U},\frac{v^*}{U},\frac{w^*}{U}\right),\quad P=\frac{P^*}{\rho U^2}. \tag{3}$$

Here ρ is the fluid density and U is a characteristic velocity.

The dimensionless governing equations become

$$\omega\frac{\partial u}{\partial s}+\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial w}{\partial \theta}+\frac{v}{r}+\varepsilon\omega[v\sin(\theta+\phi)+w\cos(\theta+\phi)]=0, \tag{4}$$

$$u=-\omega ReDa\frac{\partial P}{\partial s}, \tag{5}$$

$$v=-ReDa\frac{\partial P}{\partial r}, \tag{6}$$

$$w=-\frac{1}{r}ReDa\frac{\partial P}{\partial \theta}, \tag{7}$$

where

$$Re=Ua/\nu,\quad Da=K/a^2,\quad \varepsilon=a\kappa, \tag{8}$$

$$\omega=\frac{1}{1+\varepsilon r\sin(\theta+\phi)}. \tag{9}$$

Here κ is the curvature, and the angle ϕ is the integral with respect to s of the torsion τ.

The following transformation leads to helically symmetric solutions:

$$\theta+\phi\Rightarrow\xi,\quad \frac{\partial}{\partial s}\Rightarrow\frac{\partial}{\partial s}-\varepsilon\lambda\frac{\partial}{\partial \xi},\quad \frac{\partial}{\partial \theta}\Rightarrow\frac{\partial}{\partial \xi}, \tag{10}$$

where

$$\lambda=\tau/\kappa. \tag{11}$$

The continuity equation becomes

$$\omega\frac{\partial u}{\partial s}-\omega\varepsilon\lambda\frac{\partial u}{\partial \xi}+\frac{\partial v}{\partial r}+\frac{v}{r}+\frac{1}{r}\frac{\partial w}{\partial \xi}+\varepsilon\omega[v\sin\xi+w\cos\xi]=0. \tag{12}$$

For a fully developed flow, we set s-derivatives to zero (with the exception of the pressure derivative), so

$$\frac{\partial v}{\partial r}+\frac{v}{r}+\frac{1}{r}\frac{\partial w}{\partial \xi}+\varepsilon\omega\left[v\sin\xi+w\cos\xi-\lambda\frac{\partial u}{\partial \xi}\right]=0. \tag{13}$$

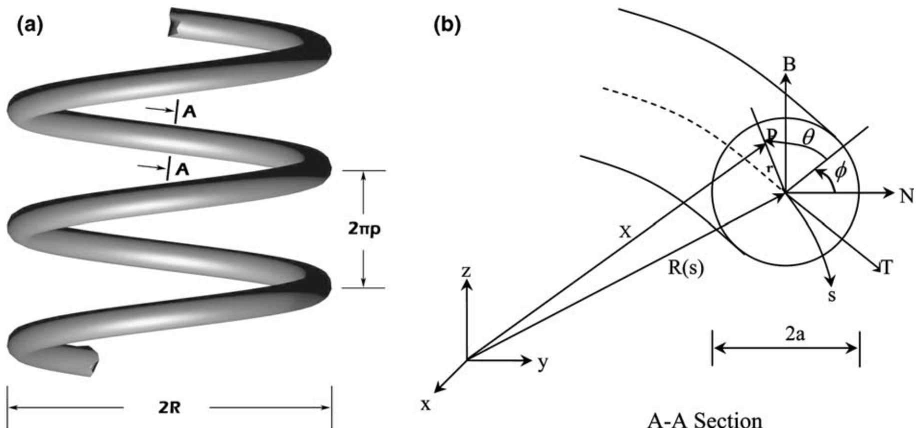

Fig. 1. (a) Helical pipe of curvature $\kappa = R/(R^2 + p^2)$ and torsion $\tau = p/(R^2 + p^2)$; (b) helical coordinate system.

The momentum equations are now:

$$u = -\omega ReDa\left(\frac{\partial P}{\partial s} - \varepsilon\lambda\frac{\partial P}{\partial \xi}\right), \tag{14}$$

$$v = -ReDa\frac{\partial P}{\partial r}, \tag{15}$$

$$w = -\frac{1}{r}ReDa\frac{\partial P}{\partial \xi}, \tag{16}$$

where

$$\omega = \frac{1}{1 + \varepsilon r\sin\xi}. \tag{17}$$

Assuming that $\varepsilon \ll 1$ (this assumes small curvature) and λ and $ReDa$ are of order unity (this implies small torsion as well) and introducing the following expansion, we let

$$u = u_0(r) + \varepsilon u_1(\xi, r) + \varepsilon^2 u_2(\xi, r) + \cdots, \tag{18}$$

$$v = \varepsilon v_1(\xi, r) + \varepsilon^2 v_2(\xi, r) + \cdots, \tag{19}$$

$$w = \varepsilon w_1(\xi, r) + \varepsilon^2 w_2(\xi, r) + \cdots, \tag{20}$$

$$P = P_0(s) + \varepsilon P_1(\xi, r) + \varepsilon^2 P_2(\xi, r) + \cdots. \tag{21}$$

The following solution for the zero-order velocity is obtained:

$$u_0 = -ReDa\frac{\partial P_0}{\partial s}, \quad v_0 = 0, \quad w_0 = 0. \tag{22}$$

Eq. (22) tells us that the zero-order velocity is a slug flow.

The following equations for the first-order variables are then obtained:

$$\frac{\partial v_1}{\partial r} + \frac{1}{r}\frac{\partial w_1}{\partial \xi} + \frac{v_1}{r} = 0, \tag{23}$$

$$u_1 = -ReDa\left[\frac{\partial P_1}{\partial s} - r\sin\xi\frac{\partial P_0}{\partial s}\right], \tag{24}$$

$$v_1 = -ReDa\frac{\partial P_1}{\partial r}, \tag{25}$$

$$w_1 = -\frac{1}{r}ReDa\frac{\partial P_1}{\partial \xi}. \tag{26}$$

The requirement that the velocity be bounded at $r = 0$ requires that $\partial P_1/\partial\xi = 0$ and so $w_1 = 0$, from Eq. (26). Then Eq. (23) implies that rv_1 is a function of s and ξ only. Again, the requirement that the velocity be bounded at $r = 0$ requires that $v_1 = 0$ and so $\partial P_1/\partial r = 0$, from Eq. (25). Hence P_1 is a function of s only. Integration with respect to s from one end of the channel to the other gives $P_1 = 0$. Hence the first-order solution is

$$u_1 = ReDa(r\sin\xi)(-\partial P_0/\partial s), \quad v_1 = w_1 = 0. \tag{27a, b, c}$$

Thus the effect of the curvature is to increase the axial velocity in one half of the channel and to decrease it in the other half, in accordance with expectations. The reader will note that $r\sin\xi$ is the distance from a "neutral diameter" ($\xi = 0$, in the direction of the normal vector **N**, and its extension) in a plane of cross-section.

At second order, the equations are

$$\frac{\partial v_2}{\partial r} + \frac{v_2}{r} + \frac{1}{r}\frac{\partial w_2}{\partial \xi} = -\lambda ReDar\cos\xi\frac{\partial P_0}{\partial s}, \tag{28}$$

$$u_2 = -ReDa\left[\frac{\partial P_2}{\partial s} + r^2\sin^2\xi\frac{\partial P_0}{\partial s}\right], \tag{29}$$

$$v_2 = -ReDa\frac{\partial P_2}{\partial r}, \tag{30}$$

$$w_2 = -\frac{1}{r} ReDa \frac{\partial P_2}{\partial \xi}. \tag{31}$$

The solution is

$$\begin{aligned} u_2 &= ReDar^2\sin^2\xi(-\partial P_0/\partial s), \\ v_2 &= \tfrac{1}{2}\lambda ReDar\cos\xi(-\partial P_0/\partial s), \\ w_2 &= 0. \end{aligned} \tag{32a, b, c}$$

It follows that, to second order,

$$u = ReDa(-\partial P_0/\partial s)(1 + \varepsilon r\sin\xi + \varepsilon^2 r^2\sin^2\xi). \tag{33}$$

The mean velocity is

$$\begin{aligned} U &= \frac{1}{\pi}\int_0^1 \int_0^{2\pi} ReDa(-\partial P_0/\partial s) \\ &\quad \times (1+\varepsilon r\sin\xi+\varepsilon^2 r^2\sin^2\xi) r\,\mathrm{d}r\,\mathrm{d}\xi \\ &= ReDa(-\partial P_0/\partial s)(1+\varepsilon^2/4). \end{aligned} \tag{34}$$

We introduce a new dimensionless variable

$$\hat{u} = u/U. \tag{35}$$

It then follows that

$$\hat{u} = 1 + \varepsilon r\sin\xi + \varepsilon^2(r^2\sin^2\xi - 1/4). \tag{36}$$

We also introduce a non-dimensional temperature $\widehat{T}$ and a Nusselt number Nu defined by

$$\widehat{T} = \frac{T^* - T_w^*}{T_m^* - T_w^*}, \tag{37}$$

$$Nu = \frac{2aq''}{k(T_w^* - T_m^*)}. \tag{38}$$

Here q'' is the wall heat flux, T_w^* is the wall temperature and T_m^* is the bulk mean temperature defined by

$$T_m^* = \frac{1}{\pi a^2 U}\int_0^a \int_0^{2\pi} u^* T^* r^*\,\mathrm{d}r^*\,\mathrm{d}\xi. \tag{39}$$

In terms of dimensional quantities, the thermal energy equation

$$u^* \frac{\partial T^*}{\partial x^*} = \frac{k}{\rho c_p}\nabla^2 T^*. \tag{40}$$

2.1. Case A. Uniform flux boundaries

For the case of boundaries held at constant uniform heat flux, the first law of thermodynamics implies that the axial temperature gradient is a constant, given by

$$\frac{\partial T^*}{\partial x^*} = \frac{2q''}{\rho c_p a U}. \tag{41}$$

We assume that the appropriate Péclet number is sufficiently large so that the axial conduction may be neglected. In terms of the helical coordinates and dimensionless quantities the thermal energy equation becomes

$$\begin{aligned} \frac{\partial^2\widehat{T}}{\partial r^2} + \frac{1}{r}\frac{\partial\widehat{T}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\widehat{T}}{\partial\xi^2} &= -Nu\hat{u} \\ &= -Nu(1+\varepsilon r\sin\xi \\ &\quad + \varepsilon^2 r^2\sin^2\xi - 1/4). \end{aligned} \tag{42}$$

This must be solved subject to the conditions that $\widehat{T} = 0$ at $r = 1$ and that $\widehat{T}$ is finite at $r = 0$. The solution is found to be

$$\begin{aligned} \widehat{T} = \frac{1}{4}Nu\Big[&1 - r^2 + \frac{1}{2}\varepsilon(r - r^3)\sin\xi \\ &- \frac{1}{24}\varepsilon^2[3 - 6r^2 + 3r^4 + 4(r^2 - r^4)\cos 2\xi]\Big]. \end{aligned} \tag{43}$$

The definitions of Nu and $\widehat{T}$ require the compatibility condition

$$\frac{1}{\pi}\int_0^1\int_0^{2\pi} \hat{u}\widehat{T} r\,\mathrm{d}r\,\mathrm{d}\xi = 1. \tag{44}$$

With Eqs. (36) and (43), Eq. (44) determines Nu. To second order in ε, one finds that

$$Nu = 8(1 + \varepsilon^2/12). \tag{45}$$

Thus we conclude that, to first order, the Nusselt number is not affected by the curvature and torsion of the pipe, but the effect of curvature is to increase the Nusselt number at second order.

2.2. Case B. Uniform temperature boundaries

In the case where the boundaries are held at constant uniform temperature, Eq. (42) is replaced by

$$\begin{aligned} &\frac{\partial^2\widehat{T}}{\partial r^2} + \frac{1}{r}\frac{\partial\widehat{T}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\widehat{T}}{\partial\xi^2} + Nu[1 + \varepsilon r\sin\xi \\ &\quad + \varepsilon^2(r^2\sin^2\xi - 1/4)]\widehat{T} = 0. \end{aligned} \tag{46}$$

We now employ a perturbation procedure. We write

$$\begin{aligned} \widehat{T} &= T_0(r) + \varepsilon T_1(r,\xi) + \varepsilon^2 T_2(r,\xi) + \cdots, \\ Nu &= Nu_0 + \varepsilon Nu_1 + \varepsilon^2 Nu_2 + \cdots. \end{aligned} \tag{47}$$

The zeroth order equation is

$$\frac{\mathrm{d}^2 T_0}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}T_0}{\mathrm{d}r} + Nu_0 T_0 = 0. \tag{48}$$

The solution, subject to the conditions $T_0(1) = 0$ and $T_0(r)$ finite at $r = 0$, is

$$T_0 = AJ_0(\tilde{\lambda}r), \tag{49}$$

where

$$Nu_0 = \tilde{\lambda}^2, \tag{50}$$

and $\tilde{\lambda} = 2.40483$ (the smallest positive zero of the function $J_0(x)$).

The compatibility condition

$$Nu = -2\frac{\mathrm{d}\widehat{T}}{\mathrm{d}r}(1) \tag{51}$$

leads to

$$A = \frac{\tilde{\lambda}}{2J_1(\tilde{\lambda})}, \tag{52}$$

$$T_0 = \frac{\tilde{\lambda}J_0(\tilde{\lambda}r)}{2J_1(\tilde{\lambda})}. \tag{53}$$

The first-order equation is

$$\frac{\partial^2 T_1}{\partial r^2} + \frac{1}{r}\frac{\partial T_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T_1}{\partial \xi^2} + Nu_0 T_1 = -(Nu_0 r \sin\xi)T_0 - Nu_1 T_0. \tag{54}$$

Solvability requires that the right-hand side of Eq. (54) be orthogonal to T_0, and this implies that $Nu_1 = 0$, since the term involving $\sin\xi$ vanishes on integration with respect to ξ from 0 to 2π.

Hence, to first order in ε, one finds that

$$Nu = (2.404830)^2 = 5.783, \tag{55}$$

independent of the value of ε. Thus we again conclude that, to this order, the Nusselt number is not affected by the curvature and torsion of the pipe.

The solution of Eq. (54) is of the form

$$T_1 = F(r)\sin\xi, \tag{56}$$

where

$$\frac{\mathrm{d}^2 F}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}F}{\mathrm{d}r} - \frac{F}{r^2} + Nu_0 F = -Nu_0 r T_0, \tag{57}$$

where T_0 is given by Eq. (53). Eq. (57) must be solved subject to the conditions that $F(1) = 0$ and F finite at $r = 0$. We have been unable to obtain an analytical solution of this equation, and we believe that none is obtainable. This blocks us from proceeding analytically. When $F(r)$ has been found numerically one can proceed to the second-order equation

$$\frac{\partial^2 T_2}{\partial r^2} + \frac{1}{r}\frac{\partial T_2}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T_2}{\partial \xi^2} + Nu_0 T_2 = -Nu_0(r\sin\xi)T_1 - Nu_0(r^2\sin^2\xi - 1/4)T_0 - Nu_2 T_0. \tag{58}$$

Again, solvability requires that the right-hand side of this equation be orthogonal to T_0. This requirement leads to the formula

$$Nu_2 = \frac{Nu_0}{4}\frac{\int_0^1\{(1-2r^2)T_0^2 - 2rFT_0\}r\,\mathrm{d}r}{\int_0^1 T_0^2 r\,\mathrm{d}r}. \tag{59}$$

The solution for $F(r)$ may be found by reduction of the second-order Eq. (57) to a system of first-order differential equations and shooting.

Define $y_1 = F$ and $y_2 = F'$ where the prime denotes a derivative with respect to the independent variable x, which replaces r in standard notation. Then we have

$$y_1' = y_2, \qquad y_2' = -\frac{y_2}{x} + \frac{y_1}{x^2} - Nu_0 y_1 - Nu_0 x T_0. \tag{60a,b}$$

One starts with $y_1(0) = \alpha$, $y_2(0) = 0$ and iterates on α until the condition $y_1(1) = 0$ is satisfied. The function y_1 thus obtained is the required function F.

In this way we found that $\alpha = 0$, a result to be expected since the non-homogeneous term in Eq. (57) vanishes when $r = 0$. Then the expression in Eq. (59) can be computed numerically. Our calculation gave $Nu_2 = 0.40762$. We conclude that

$$Nu = 5.783(1 + 0.0705\varepsilon^2). \tag{61}$$

3. Discussion

We conclude that the effect of the curvature is to increase the Nusselt number. The increase is a second-order effect, and is the result of the addition of three effects, due to respectively (i) the second-order velocity distribution, (ii) the second-order temperature distribution, and (iii) the interaction of the first-order velocity distribution with the first-order temperature distribution. It is possible to trace the individual contributions of these three effects. It turns out that in Case A (uniform flux boundaries) the contributions to Nu_2 are in the ratio (1:1:−1), and the net contribution is positive. In Case B (uniform temperature boundaries) the contributions are in the ratio (1:0:−0.5000) and again the net contribution to Nu_2 is positive.

Looking back at our analysis, we see that the torsion affects the velocity field at second order (see the term in λ in Eq. 32(b)), but this does not lead to any second-order term in the expression for the Nusselt number. The effect is to produce a small velocity component in the radial direction, which is positive in one half of the pipe (bounded by a diameter in the direction of the binormal vector **B**) and negative in the other half plane. The insignificant effect of torsion on heat transfer predicted here is consistent with the experimental result of Xin and Ebadian [7] who reported that no effects of torsion were observed in the scope of their investigation with various Newtonian fluids. (The numerical work of Yang et al. [2,5] indicates that for a fluid clear of solid material the presence of moderate torsion can decrease the Nusselt number by a significant amount, with the effect increasing as the Prandtl number increases. However, this could be due to the non-linear inertial term that is present in the momentum equation for a the clear fluid but is absent for the Darcy model.) Our result for a porous medium is also consistent with the conclusion reached by Bolinder [15] for flow in a fluid clear of solid material.

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